



EGMO | 2014

European Girls' Mathematical Olympiad

Antalya • Turkey

Problems and Solutions

Day 1

The EGMO 2014 Problem Committee thanks the following countries for submitting problem proposals:

- Bulgaria
- Iran
- Japan
- Luxembourg
- Netherlands
- Poland
- Romania
- Ukraine
- United Kingdom

The Members of the Problem Committee:

Okan Tekman

Selim Bahadır

Şahin Emrah

Fehmi Emre Kadan

1. Determine all real constants t such that whenever a, b, c are the lengths of the sides of a triangle, then so are $a^2 + bct, b^2 + cat, c^2 + abt$.

Proposed by S. Khan, UNK



The answer is the interval $[2/3, 2]$.

Solution 1.

If $t < 2/3$, take a triangle with sides $c = b = 1$ and $a = 2 - \epsilon$. Then $b^2 + cat + c^2 + abt - a^2 - bct = 3t - 2 + \epsilon(4 - 2t - \epsilon) \leq 0$ for small positive ϵ ; for instance, for any $0 < \epsilon < (2 - 3t)/(4 - 2t)$.

On the other hand, if $t > 2$, then take a triangle with sides $b = c = 1$ and $a = \epsilon$. Then $b^2 + cat + c^2 + abt - a^2 - bct = 2 - t + \epsilon(2t - \epsilon) \leq 0$ for small positive ϵ ; for instance, for any $0 < \epsilon < (t - 2)/(2t)$.

Now assume that $2/3 \leq t \leq 2$ and $b + c > a$. Then using $(b + c)^2 \geq 4bc$ we obtain

$$\begin{aligned} b^2 + cat + c^2 + abt - a^2 - bct &= (b + c)^2 + at(b + c) - (2 + t)bc - a^2 \\ &\geq (b + c)^2 + at(b + c) - \frac{1}{4}(2 + t)(b + c)^2 - a^2 \\ &\geq \frac{1}{4}(2 - t)(b + c)^2 + at(b + c) - a^2. \end{aligned}$$

As $2 - t \geq 0$ and $t > 0$, this last expression is an increasing function of $b + c$, and hence using $b + c > a$ we obtain

$$b^2 + cat + c^2 + abt - a^2 - bct > \frac{1}{4}(2 - t)a^2 + ta^2 - a^2 = \frac{3}{4}\left(t - \frac{2}{3}\right)a^2 \geq 0$$

as $t \geq 2/3$. The other two inequalities follow by symmetry.

Solution 2.

After showing that t must be in the interval $[2/3, 2]$ as in **Solution 1**, we let $x = (c + a - b)/2$, $y = (a + b - c)/2$ and $z = (b + c - a)/2$ so that $a = x + y$, $b = y + z$, $c = z + x$. Then we have:

$$b^2 + cat + c^2 + abt - a^2 - bct = (x^2 + y^2 - z^2 + xy + xz + yz)t + 2(z^2 + xz + yz - xy)$$

Since this linear function of t is positive both at $t = 2/3$ where

$$\frac{2}{3}(x^2 + y^2 - z^2 + xy + xz + yz) + 2(z^2 + xz + yz - xy) = \frac{2}{3}((x - y)^2 + 4(x + y)z + 2z^2) > 0$$

and at $t = 2$ where

$$2(x^2 + y^2 - z^2 + xy + xz - yz) + 2(z^2 + xz + yz + xy) = 2(x^2 + y^2) + 4(x + y)z > 0,$$

it is positive on the entire interval $[2/3, 2]$.

Solution 3.

After the point in **Solution 2** where we obtain

$$b^2 + cat + c^2 + abt - a^2 - bct = (x^2 + y^2 - z^2 + xy + xz + yz)t + 2(z^2 + xz + yz - xy)$$

we observe that the right hand side can be rewritten as

$$(2 - t)z^2 + (x - y)^2t + (3t - 2)xy + z(x + y)(2 + t).$$

As the first three terms are non-negative and the last term is positive, the result follows.

Solution 4.

First we show that t must be in the interval $[2/3, 2]$ as in **Solution 1**. Then:

Case 1: If $a \geq b, c$, then $ab + ac - bc > 0$, $2(b^2 + c^2) \geq (b + c)^2 > a^2$ and $t \geq 2/3$ implies:

$$\begin{aligned} b^2 + cat + c^2 + abt - a^2 - bct &= b^2 + c^2 - a^2 + (ab + ac - bc)t \\ &\geq (b^2 + c^2 - a^2) + \frac{2}{3}(ab + ac - bc) \\ &\geq \frac{1}{3}(3b^2 + 3c^2 - 3a^2 + 2ab + 2ac - 2bc) \\ &\geq \frac{1}{3}[(2b^2 + 2c^2 - a^2) + (b - c)^2 + 2a(b + c - a)] \\ &> 0 \end{aligned}$$

Case 2: If $b \geq a, c$, then $b^2 + c^2 - a^2 > 0$. If also $ab + ac - bc \geq 0$, then $b^2 + cat + c^2 + abt - a^2 - bct > 0$. If, on the other hand, $ab + ac - bc \leq 0$, then since $t \leq 2$, we have:

$$\begin{aligned} b^2 + cat + c^2 + abt - a^2 - bct &\geq b^2 + c^2 - a^2 + 2(ab + ac - bc) \\ &\geq (b - c)^2 + a(b + c - a) + a(b + c) \\ &> 0 \end{aligned}$$

By symmetry, we are done.

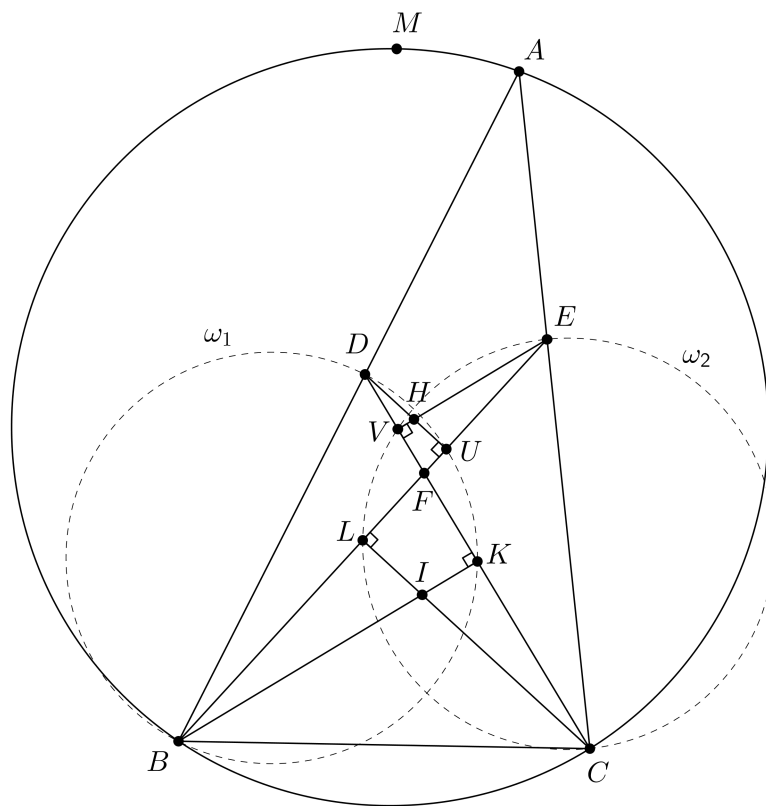
2. Let D and E be two points on the sides AB and AC , respectively, of a triangle ABC , such that $DB = BC = CE$, and let F be the point of intersection of the lines CD and BE . Prove that the incenter I of the triangle ABC , the orthocenter H of the triangle DEF and the midpoint M of the arc BAC of the circumcircle of the triangle ABC are collinear.

Proposed by Danylo Khilko, UKR



Solution 1.

As $DB = BC = CE$ we have $BI \perp CD$ and $CI \perp BE$. Hence I is orthocenter of triangle BFC . Let K be the point of intersection of the lines BI and CD , and let L be the point of intersection of the lines CI and BE . Then we have the power relation $IB \cdot IK = IC \cdot IL$. Let U and V be the feet of the perpendiculars from D to EF and E to DF , respectively. Now we have the power relation $DH \cdot HU = EH \cdot HV$.



Let ω_1 and ω_2 be the circles with diameters BD and CE , respectively. From the power relations above we conclude that IH is the radical axis of the circles ω_1 and ω_2 .

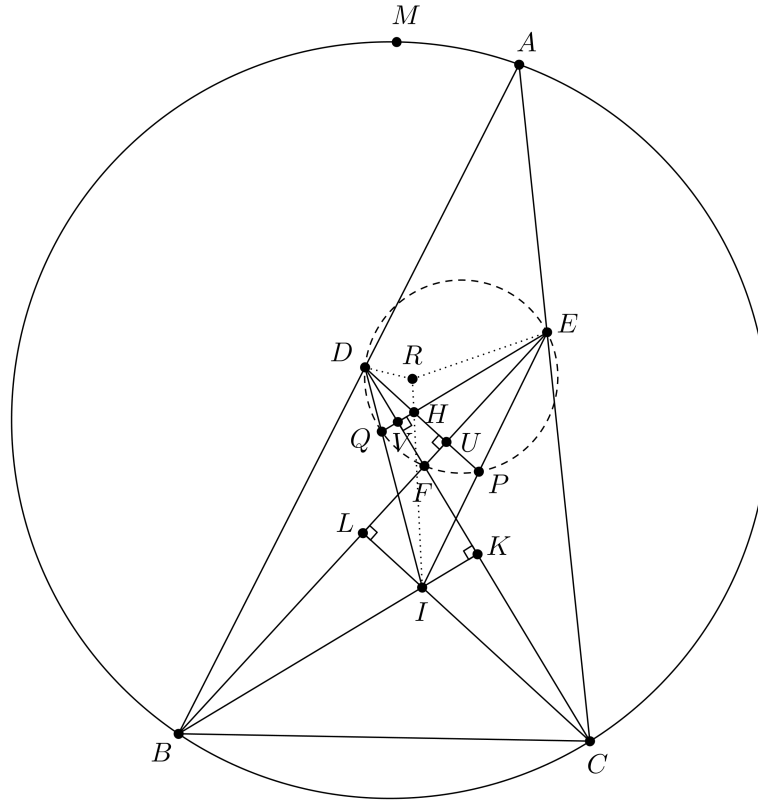
Let O_1 and O_2 be centers of ω_1 and ω_2 , respectively. Then $MB = MC$, $BO_1 = CO_2$ and $\angle MBO_1 = \angle MCO_2$, and the triangles MBO_1 and MCO_2 are congruent. Hence $MO_1 = MO_2$. Since radii of ω_1 and ω_2 are equal, this implies that M lies on the radical axis of ω_1 and ω_2 and M, I, H are collinear.

Solution 2.

Let the points K, L, U, V be as in **Solution 1**. Let P be the point of intersection of DU and EI , and let Q be the point of intersection of EV and DI .

Since $DB = BC = CE$, the points CI and BI are perpendicular to BE and CD , respectively. Hence the lines BI and EV are parallel and $\angle IEB = \angle IBE = \angle UEH$. Similarly, the lines CI and DU are parallel and $\angle IDC = \angle ICD = \angle VDH$. Since $\angle UEH = \angle VDH$, the points D, Q, F, P, E are concyclic. Hence $IP \cdot IE = IQ \cdot ID$.

Let R be the second point intersection of the circumcircle of triangle HEP and the line HI . As $IH \cdot IR = IP \cdot IE = IQ \cdot ID$, the points D, Q, H, R are also concyclic. We have $\angle DQH = \angle EPH = \angle DFE = \angle BFC = 180^\circ - \angle BIC = 90^\circ - \angle BAC/2$. Now using the concyclicity of D, Q, H, R , and E, P, H, R we obtain $\angle DRH = \angle ERH = 180^\circ - (90^\circ - \angle BAC/2) = 90^\circ + \angle BAC/2$. Hence R is inside the triangle DEH and $\angle DRE = 360^\circ - \angle DRH - \angle ERH = 180^\circ - \angle BAC$ and it follows that the points A, D, R, E are concyclic.



As $MB = MC$, $BD = CE$, $\angle MBD = \angle MCE$, the triangles MBD and MCE are congruent and $\angle MDA = \angle MEA$. Hence the points M, D, E, A are concyclic. Therefore the points M, D, R, E, A are concyclic. Now we have $\angle MRE = 180^\circ - \angle MAE = 180^\circ - (90^\circ + \angle BAC/2) = 90^\circ - \angle BAC/2$ and since $\angle ERH = 90^\circ + \angle BAC/2$, we conclude that the points I, H, R, M are collinear.

Solution 3.

Suppose that we have a coordinate system and $(b_x, b_y), (c_x, c_y), (d_x, d_y), (e_x, e_y)$ are the coordinates of the points B, C, D, E , respectively. From $\overrightarrow{BI} \cdot \overrightarrow{CD} = 0, \overrightarrow{CI} \cdot \overrightarrow{BE} = 0, \overrightarrow{EH} \cdot \overrightarrow{CD} = 0, \overrightarrow{DH} \cdot \overrightarrow{BE} = 0$ we obtain $\overrightarrow{IH} \cdot (\overrightarrow{B} - \overrightarrow{C} - \overrightarrow{E} + \overrightarrow{D}) = 0$. Hence the slope of the line IH is $(c_x + e_x - b_x - d_x)/(b_y + d_y - c_y - e_y)$.

Assume that the x -axis lies along the line BC , and let $\alpha = \angle BAC, \beta = \angle ABC, \theta = \angle ACB$. Since $DB = BC = CE$, we have $c_x - b_x = BC, e_x - d_x = BC - BC \cos \beta - BC \cos \theta, b_y = c_y = 0, d_y - e_y = BC \sin \beta - BC \sin \theta$. Therefore the slope of IH is $(2 - \cos \beta - \cos \theta)/(\sin \beta - \sin \theta)$.

Now we will show that the slope of the line MI is the same. Let r and R be the inradius and circumradius of the triangle ABC , respectively. As $\angle BMC = \angle BAC = \alpha$ and $BM = MC$, we have

$$m_y - i_y = \frac{BC}{2} \cot\left(\frac{\alpha}{2}\right) - r \text{ and } m_x - i_x = \frac{AC - AB}{2}$$

where (m_x, m_y) and (i_x, i_y) are the coordinates of M and I , respectively. Therefore the slope of MI is $(BC \cot(\alpha/2) - 2r)/(AC - AB)$.

Now the equality of these slopes follows using

$$\frac{BC}{\sin \alpha} = \frac{AC}{\sin \beta} = \frac{AB}{\sin \theta} = 2R,$$

hence

$$BC \cot\left(\frac{\alpha}{2}\right) = 4R \cos^2\left(\frac{\alpha}{2}\right) = 2R(1 + \cos \alpha)$$

and

$$\frac{r}{R} = \cos \alpha + \cos \beta + \cos \theta - 1$$

as

$$\frac{BC \cot(\alpha/2) - 2r}{AC - AB} = \frac{2R(1 + \cos \alpha) - 2r}{2R(\sin \beta - \sin \theta)} = \frac{2 - \cos \beta - \cos \theta}{\sin \beta - \sin \theta}$$

giving the collinearity of the points I, H, M .

Solution 4.

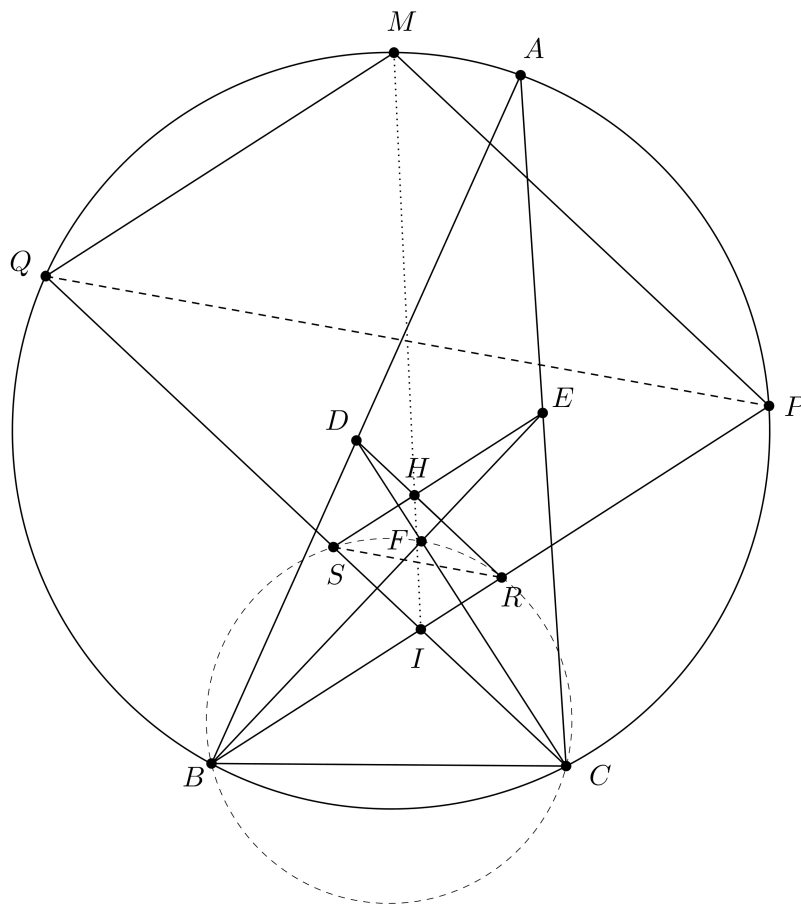
Let the bisectors BI and CI meet the circumcircle of ABC again at P and Q , respectively. Let the altitude of DEF belonging to D meet BI at R and the one belonging to E meet CI at S .

Since BI is angle bisector of the isosceles triangle CBD , BI and CD are perpendicular. Since EH and DF are also perpendicular, HS and RI are parallel. Similarly, HR and SI are parallel, and hence $HSIR$ is a parallelogram.

On the other hand, as M is the midpoint of the arc BAC , we have $\angle MPI = \angle MPB = \angle MQC = \angle MQI$, and $\angle PIQ = (\widehat{PA} + \widehat{CB} + \widehat{AQ})/2 = (\widehat{PC} + \widehat{CB} + \widehat{BQ})/2 = \angle PMQ$. Therefore $MPIQ$ is a parallelogram.

Since CI is angle bisector of the isosceles triangle BCE , the triangle BSE is also isosceles. Hence $\angle FBS = \angle EBS = \angle SEB = \angle HEF = \angle HDF = \angle RDF = \angle FCS$ and B, S, F, C are concyclic. Similarly, B, F, R, C are concyclic. Therefore B, S, R, C are concyclic. As B, Q, P, C are also concyclic, SR and QP are parallel.

Now it follows that $HSIR$ and $MQIP$ are homothetic parallelograms, and therefore M, H, I are collinear.



3. We denote the number of positive divisors of a positive integer m by $d(m)$ and the number of distinct prime divisors of m by $\omega(m)$. Let k be a positive integer. Prove that there exist infinitely many positive integers n such that $\omega(n) = k$ and $d(n)$ does not divide $d(a^2 + b^2)$ for any positive integers a, b satisfying $a + b = n$.

Proposed by JPN



Solution.

We will show that any number of the form $n = 2^{p-1}m$ where m is a positive integer that has exactly $k - 1$ prime factors all of which are greater than 3 and p is a prime number such that $(5/4)^{(p-1)/2} > m$ satisfies the given condition.

Suppose that a and b are positive integers such that $a + b = n$ and $d(n) \mid d(a^2 + b^2)$. Then $p \mid d(a^2 + b^2)$. Hence $a^2 + b^2 = q^{cp-1}r$ where q is a prime, c is a positive integer and r is a positive integer not divisible by q . If $q \geq 5$, then

$$2^{2p-2}m^2 = n^2 = (a + b)^2 > a^2 + b^2 = q^{cp-1}r \geq q^{p-1} \geq 5^{p-1}$$

gives a contradiction. So q is 2 or 3.

If $q = 3$, then $a^2 + b^2$ is divisible by 3 and this implies that both a and b are divisible by 3. This means $n = a + b$ is divisible by 3, a contradiction. Hence $q = 2$.

Now we have $a + b = 2^{p-1}m$ and $a^2 + b^2 = 2^{cp-1}r$. If the highest powers of 2 dividing a and b are different, then $a + b = 2^{p-1}m$ implies that the smaller one must be 2^{p-1} and this makes 2^{2p-2} the highest power of 2 dividing $a^2 + b^2 = 2^{cp-1}r$, or equivalently, $cp - 1 = 2p - 2$, which is not possible. Therefore $a = 2^t a_0$ and $b = 2^t b_0$ for some positive integer $t < p - 1$ and odd integers a_0 and b_0 . Then $a_0^2 + b_0^2 = 2^{cp-1-2t}r$. The left side of this equality is congruent to 2 modulo 4, therefore $cp - 1 - 2t$ must be 1. But then $t < p - 1$ gives $(c/2)p = t + 1 < p$, which is not possible either.